

# Josephson junctions in narrow thin-film strips

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(Dated: February 20, 2008)

We study the field dependence of the maximum current  $I_m(H)$  in narrow edge-type thin-film Josephson junctions. The junction extends across thin-film strip of width  $W \ll \Lambda = 2\lambda^2/d$ , the London depth  $\lambda \gg d$ ,  $d$  is the film thickness. We calculate  $I_m(H)$  within nonlocal Josephson electrodynamics, which takes into account the stray fields affecting tunneling currents. For  $W \ll c\phi_0/8\pi^2\Lambda g_c$ ,  $g_c$  is the critical sheet current density, the phase difference along the junction depends only on the junction geometry and the applied field, but is independent of the  $g_c$ , i.e., it is universal. Zeros of  $I_m(H)$  are equidistant only in large fields (unlike the case of junctions with bulk banks); they are spaced by  $\sim \phi_0/W^2$  that is much smaller than  $\phi_0/W\lambda$  of bulk junctions. The maxima of  $I_m(H)$  decrease as  $1/\sqrt{H}$ , slower than  $1/H$  for the bulk.

PACS numbers: 74.60. Ec, 74.60. Ge

Keywords: maximum supercurrent, Josephson junction, narrow Josephson junction, thin films

The physics of the edge-type thin-film Josephson junctions (e.g., two films in the  $(x, y)$  plane touching only along the edges at  $x = 0$  with no overlap) differs from that of the junctions with bulk banks mainly because of the stray fields, that affect the currents in the junction and in the thin-film banks. The phase difference  $\varphi$  across the junction is also affected by the stray fields. As a result,  $\varphi$  is described by an integral equation, i.e., the problem becomes *nonlocal* [1, 2, 3, 4, 5, 6].

Development of nonlocal electrodynamics of such junctions is still in progress and is a subject of growing interest [7]. Long-range stray-fields are relevant for physics of sequences of interchanging 0- and  $\pi$ -junctions [8, 9, 10, 11, 12, 13, 14]. These anomalous chains of tunnel Josephson junctions are studied also for the thin-film superconductor-ferromagnet-superconductor heterostructures [11, 12], asymmetric grain boundaries in  $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}$  [10, 13], and  $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}/\text{Nb}$  zigzag junctions [14].

The phase distribution  $\varphi(y)$  along thin-film edge-type junctions has a length scale  $\ell = c\phi_0/8\pi^2\Lambda g_c$ , the thin-film analog of the Josephson length [6];  $g_c$  is the critical sheet current density,  $\Lambda = 2\lambda^2/d$ ,  $\lambda \gg d$  is the London penetration depth, and  $d$  is the film thickness. We show in this work that when the width  $W$  of the junction containing strip (and the junction length that are the same) is less than  $\ell$ , the distribution of the phase difference  $\varphi(y)$  becomes  $\ell$  independent, i.e., the same for junctions with different critical currents. In other words, for  $W \ll \ell$ ,  $\varphi(y)$  is a universal function, that depends only on the applied field and the junction geometry.

In this situation, we evaluate the field dependence of the maximum supercurrent  $I_m(H)$  through the junction that turns out quite different from the standard Fraunhofer pattern of bulk junctions. Zeros of  $I_m(H)$  become equidistant only in large fields unlike in bulk junctions, and are separated by  $\Delta H \sim \phi_0/W^2$ , which is much

smaller than  $\phi_0/W\lambda$  of bulk junctions of the same length. The maxima of  $I_m(H)$  decrease as  $1/\sqrt{H}$ , that is significantly slower than  $1/H$  for the bulk. We show that  $I_m(H)$  for a SQUID made of narrow thin-film strips with edge-type Josephson junctions differs remarkably from the canonic pattern of the bulk junctions.

Let the  $x$  axis be along the strip and  $z$  be perpendicular to the film; the junction is located at  $x = 0$ ,  $0 \leq y \leq W$ . The sheet current density  $\mathbf{g} = (g_x, g_y)$  can always be written as  $\mathbf{g} = \text{curl } S \hat{\mathbf{z}} = (\partial_y S, -\partial_x S)$ , where the  $S(x, y)$  is the stream function [6]. Since the current component normal to the edges is zero,  $S$  is constant along the edges ( $y = 0, W$ ) and the total current through the strip is

$$I = \int_0^W dy g_x = \int_0^W dy \partial_y S(0, y) = S(W) - S(0). \quad (1)$$

The London equation integrated over the film thickness reads:

$$h_z + \frac{2\pi\Lambda}{c} \text{curl}_z \mathbf{g} = \frac{\phi_0}{2\pi} \delta(x) \varphi'(y), \quad (2)$$

where  $h_z$  consists of the applied field  $H$  and the part related to  $\mathbf{g}$  by the Biot-Savart integral. The right-hand side here is a manifestation of a general rule: the field of a Josephson junction is equivalent to the field of a set of vortices distributed along the junction with the line density  $\varphi'(y)/2\pi$  [6, 15].

In narrow strips, the self-field due to the current  $\mathbf{g}$  is of the order  $g/c$ , whereas the second term on the left-hand side of Eq. (2) is of the order  $g\Lambda/cW \gg g/c$ . Hence, the self-field can be disregarded, unlike the *applied* field  $H$ . Substituting  $\text{curl}_z \mathbf{g} = -\nabla^2 S$  in Eq. (2), one obtains:

$$\frac{2\pi\Lambda}{c} \nabla^2 S = -\frac{\phi_0}{2\pi} \delta(x) \varphi'(y) + H. \quad (3)$$

This *linear* equation has solutions  $S = S_1 + S_2$  such that

$$\frac{2\pi\Lambda}{c\phi_0} \nabla^2 S_1 = -\frac{\delta(x)}{2\pi} \varphi'(y), \quad (4)$$

$$\frac{2\pi\Lambda}{c\phi_0} \nabla^2 S_2 = \frac{H}{\phi_0}. \quad (5)$$

The boundary condition (1) is satisfied if  $S_1(W) = S_1(0) = 0$  and  $S_2(W) - S_2(0) = I$ . Hence we have:

$$S_1(\mathbf{r}) = \int d\boldsymbol{\rho} \delta(u) \frac{\varphi'(v)}{2\pi} G(\mathbf{r}, \boldsymbol{\rho}), \quad (6)$$

$$S_2 = \frac{cH}{4\pi\Lambda} y(y - W) + \frac{I}{W} y. \quad (7)$$

Here,  $\mathbf{r} = (x, y)$  and  $\boldsymbol{\rho} = (u, v)$ ;  $G(\mathbf{r}, \boldsymbol{\rho})$  is the Green's function for Eq. (4) with zero boundary conditions that satisfies  $(2\pi\Lambda/c\phi_0) \nabla^2 G = -\delta(\mathbf{r} - \boldsymbol{\rho})$ , an equation well studied in electrostatics [16]:

$$\mathcal{G}(\mathbf{r}, \boldsymbol{\rho}) = \tanh^{-1} \frac{\sin V \sin Y}{\cosh(X - U) - \cos Y \cos V}; \quad (8)$$

$\mathcal{G} = 4\pi^2\Lambda G/c\phi_0$ , the capitals stand for corresponding coordinates in units of  $W/\pi$ . The Green's function  $G(\mathbf{r}, \boldsymbol{\rho})$  gives in fact the current distribution of a single vortex at  $\mathbf{r} = \boldsymbol{\rho}$ .

Clearly,  $S_1$  describes the current perturbation due to the junction. The first term in  $S_2$  represents the screening currents due to the applied field, whereas the second is due to the field of a uniform transport current.

Given the stream function, we obtain the sheet current density through the junction:

$$g_c \sin \varphi(y) = g_x(0, y) = \partial_y S(0, y) = \int_0^W dv \frac{\varphi'(v)}{2\pi} \partial_y G(0, y, 0, v) + \frac{cH}{2\pi\Lambda} \left( y - \frac{W}{2} \right) + \frac{I}{W}. \quad (9)$$

We rewrite this integral equation for the phase  $\varphi(y)$  as:

$$\frac{W}{\ell} \sin \varphi = \int_0^\pi \frac{dV \varphi'(V) \sin V}{\cos Y - \cos V} + h \left( Y - \frac{\pi}{2} \right) + i, \quad (10)$$

where

$$\ell = \frac{c\phi_0}{8\pi^2\Lambda g_c}, \quad h = \frac{4W^2}{\phi_0} H, \quad i = \frac{8\pi^2\Lambda}{c\phi_0} I \quad (11)$$

are the characteristic length, the reduced field, and the reduced current.

To establish the boundary conditions for  $\varphi(y)$  we employ the London equation for  $g_y(\pm 0, y)$  on the two junction banks

$$g_y(\pm 0, y) = -\frac{c\phi_0}{4\pi^2\Lambda} \left[ \frac{\partial \chi(\pm 0, y)}{\partial y} - \frac{2\pi}{\phi_0} A_y \right], \quad (12)$$

where  $\chi(x, y)$  is the phase and  $\mathbf{A}$  is the vector potential. We subtract these equations and utilize the continuity of  $\mathbf{A}$  to obtain  $\varphi'(y) \propto g_y(0, y)$ . The current  $g_y$  must vanish at the junction edges, i.e.,

$$\varphi'(0) = \varphi'(W) = 0. \quad (13)$$

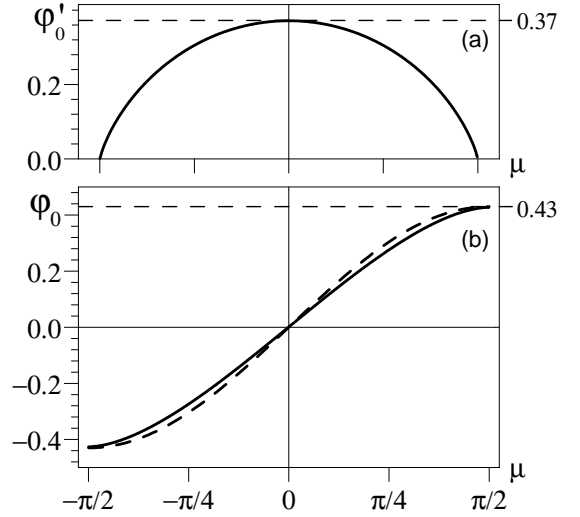


FIG. 1: (a) The function  $\varphi'_0(\mu)$  calculated according to Eq. (16). (b) The solid line is  $\varphi_0(\mu)$  obtained by numerical integration of  $\varphi'_0(\mu)$  shown in the panel (a). The dashed line is the approximation  $\varphi_0(\mu) = 0.43 \sin \mu$ .

We note that the length  $\ell$  along with  $g_c$ , the only material parameter of the junction, enters only the left-hand side of Eq. (10). In narrow junctions with  $W \ll \ell$ , this term can be disregarded. While neglecting the term  $\propto W/\ell$  we have to disregard also the transport current  $i$ ; otherwise, integrating both sides of Eq. (10) over the strip does not produce identity.

The truncated Eq. (10) reveals a remarkable feature of junctions in narrow strips: the phase is just proportional to the applied field and can be written as  $\varphi(y) = h \varphi_0(y)$  where  $\varphi_0(y)$  is an universal function governed by an integral equation

$$\int_0^\pi dV \frac{\varphi'_0(V) \sin V}{\cos Y - \cos V} + Y - \frac{\pi}{2} = 0, \quad (14)$$

which does not contain  $g_c$ , the physical parameter of the junction quality. To study this function, we introduce  $s = \cos V$ ,  $t = \cos Y$  and write Eq. (14) in the form:

$$\frac{1}{2\pi} \int_{-1}^1 \frac{J(s) ds}{t - s} = B_n(t), \quad (15)$$

$$J = 2\pi \sqrt{1 - s^2} \frac{d\varphi_0}{ds}, \quad B_n = -\sin^{-1} t.$$

The reason for this manipulation is this: Eq. (15) is the Biot-Savart expression for the normal component of the “field”  $B_n$  at the surface of a thin strip  $-1 < s < 1$  carrying the “sheet current”  $J(s)$ . This integral equation can be inverted [17]. One, however, should have in mind that the current  $J(s)$  is not determined uniquely by one field component; currents of the form  $C/\sqrt{1 - s^2}$  with an arbitrary constant  $C$  correspond to full Meissner screening and to zero normal component of the “field”. The

latter flexibility allows us to obtain the solution  $\varphi'_0(V)$  of Eq. (14) that satisfies the boundary conditions (13):

$$\varphi'_0(\mu) = \frac{1}{\pi^2 \cos \mu} \left( 2 - \int_{-\pi/2}^{\pi/2} \frac{\eta \cos^2 \eta d\eta}{\sin \mu - \sin \eta} \right), \quad (16)$$

where the origin is shifted to the strip middle for convenience,  $\mu = Y - \pi/2$ .

The integral in Eq. (16) is understood as Cauchy principal value and can be done numerically. The universal function  $\varphi'_0(\mu)$  so calculated is shown in Fig. 1 (a). The result of the numerical integration of this function obtained requiring  $\varphi_0(\mu)$  to be an odd function of  $\mu$  is shown in Fig. 1 (b). In particular, this calculation gives  $\varphi_0(\pi/2) - \varphi_0(-\pi/2) \approx 0.86$ .

Thus, for any applied field in narrow thin-film junctions the phase  $\varphi(\mu)$  takes the form  $\varphi(\mu) = h\varphi_0(\mu) + \theta$ , where  $\theta$  is a constant. The total current through the junction is

$$I = \frac{g_c W}{\pi} \int_{-\pi/2}^{\pi/2} d\mu \sin[h\varphi_0(\mu) + \theta]. \quad (17)$$

Maximizing this with respect to  $\theta$  provides  $\theta = \pi/2$  and the maximum current  $I_m$ :

$$\frac{I_m}{g_c W} = \frac{1}{\pi} \left| \int_{-\pi/2}^{\pi/2} d\mu \cos[h\varphi_0(\mu)] \right|. \quad (18)$$

Hence,  $I_m(H)$  can be evaluated numerically; a good approximation for  $I_m(H)$  can be obtained as follows:

The odd function  $\varphi_0(\mu)$  can be written as the Fourier series  $\sum a_n \sin(2n+1)\mu$  to satisfy the boundary conditions (13). We take the lowest approximant  $\varphi_0 = a_0 \sin \mu$  with  $a_0 = 0.43$  to fit the difference  $\varphi_0(W) - \varphi_0(0) = 0.86$  that is found integrating numerically the exact derivative in Eq. (16). The comparison of the phase found numerically with  $a_0 \sin \mu$  is shown in Fig. 1 (b).

In this approximation we have:

$$\frac{I_m}{g_c W} = \frac{1}{\pi} \left| \int_{-\pi/2}^{\pi/2} d\mu \cos(h a_0 \sin \mu) \right| = |J_0(a_0 h)|. \quad (19)$$

Figure 2 shows that this approximation is quite accurate as compared to  $I_m(H)$  calculated numerically with the help of Eq. (18). Zeros of the Bessel function  $J_0(x)$  are equidistant for large arguments, but they are spaced roughly by  $\pi$  everywhere. Hence zeros of  $I_m(h)$  are separated by  $a_0 \Delta h \simeq \pi$ , or in common units by:

$$\Delta H \simeq 1.8 \frac{\phi_0}{W^2}. \quad (20)$$

It is worth recalling that in bulk junctions of the length  $W$  the zeros are separated by  $\Delta H \approx 2\phi_0/W\lambda$  that exceeds by much the thin-film spacing.

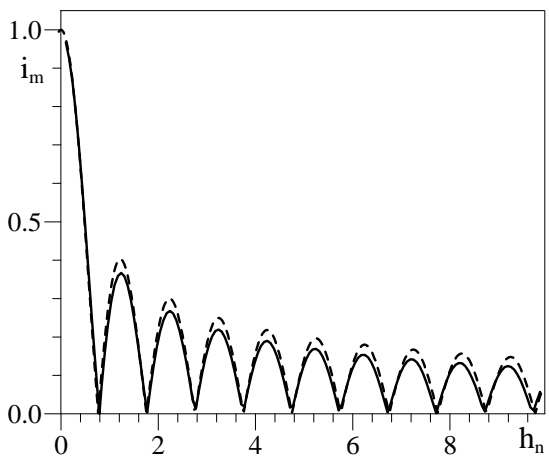


FIG. 2: The maximum supercurrent  $i_m = I_m/g_c W$  versus the normalized applied field  $h_n = 4a_0 W^2 H/\pi\phi_0$ . The dashed line is the approximation (19).

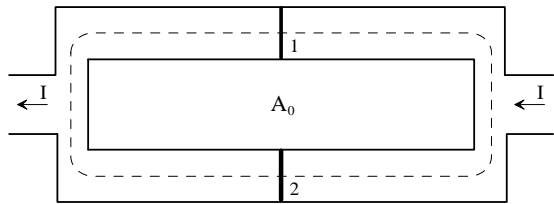


FIG. 3: Sketch of a rectangular SQUID made of two narrow thin-film strips with identical edge-type junctions 1 and 2.

In the high-field region one can use the large argument asymptotics of  $J_0$  to obtain:

$$I_m \approx 0.61 g_c \sqrt{\frac{\phi_0}{H}} \left| \cos \left( 1.72 \frac{HW^2}{\phi_0} - \frac{\pi}{4} \right) \right|. \quad (21)$$

Thus, the maxima of  $I_m(H)$  decrease as  $1/\sqrt{H}$ , i.e., slower than in the bulk case where  $I_m \propto 1/H$ .

It is worth noting that in high fields the maxima  $I_m(H)$  do not depend on the junction length  $W$ . Qualitatively, this comes about because the tunneling current  $g_x = g_c \sin(h\varphi_0 + \theta)$  oscillates fast for  $h \gg 1$  so that most of the junction length does not contribute to the total current, unlike currents in narrow belts of the width  $\delta \simeq 0.3\sqrt{\phi_0/H}$  near the strip edges.

Let us consider now current flowing through rectangular SQUID (superconducting quantum interference device) made of narrow thin-film strips with two identical Josephson junctions sketched in Fig. 3. In zero field the current distribution is symmetric with respect to the SQUID center and the line integral of  $\mathbf{g}$  along any symmetric contour is zero. When the field is applied, this symmetry is violated by the screening currents. However, at the contour in the strips middle (shown in the figure) the screening currents vanish so that the contour integral of  $\mathbf{g}$  remains zero. This contour crosses the junc-

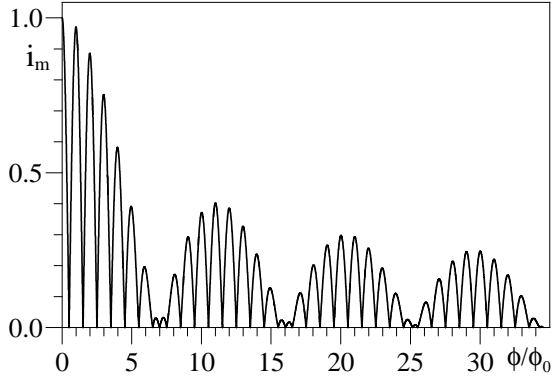


FIG. 4: The maximum supercurrent  $i_m = I_m/2g_cW$  versus flux  $\phi/\phi_0$  for a rectangular SQUID (Fig. 3) with  $A_0/W^2 = 5$ .

tions at their middle, where the local coordinates  $\mu = 0$ . Clearly, the flux  $\phi$  enclosed by this contour does not change if the contour is shifted as a whole by  $\mu$ . Integrating  $\mathbf{g} = -(c\phi_0/4\pi^2\Lambda)(\nabla\chi + 2\pi\mathbf{A}/\phi_0)$  over such a contour we obtain:

$$\varphi_2(\mu) - \varphi_1(\mu) = 2\pi \frac{\phi}{\phi_0}. \quad (22)$$

The total current through the system is given by:

$$\begin{aligned} \frac{\pi I}{g_c W} &= \int_{-\pi/2}^{\pi/2} d\mu (\sin \varphi_1 + \sin \varphi_2) \\ &= \int_{-\pi/2}^{\pi/2} d\mu \left[ \sin(h\varphi_0 + \theta) + \sin\left(h\varphi_0 + \theta + \frac{2\pi\phi}{\phi_0}\right) \right] \\ &= 2 \int_{-\pi/2}^{\pi/2} d\mu \sin\left(h\varphi_0 + \theta + \frac{\pi\phi}{\phi_0}\right) \cos\left(\frac{\pi\phi}{\phi_0}\right). \end{aligned} \quad (23)$$

As above,  $\theta$  is a constant with respect to which the current should be maximized. The maximum current then corresponds to  $\theta = \pi/2 - \pi\phi/\phi_0$ :

$$I_m = 2g_cW \left| J_0 \left( 4a_0 \frac{W^2}{A_0} \frac{\phi}{\phi_0} \right) \cos \left( \pi \frac{\phi}{\phi_0} \right) \right|, \quad (24)$$

where  $A_0$  is the area of the “central” contour. Note that our argument is valid if the SQUID hole area is large relative to the area of superconducting branches. In this case the difference between the flux enclosed by the “central” contour and the SQUID hole area can be disregarded.

Thus, the standard SQUID pattern given by  $|\cos(\pi\phi/\phi_0)|$  is modulated in our case by a slowly varying Bessel function. An example of  $I_m(\phi/\phi_0)$  is shown in Fig. 4 for a rectangular SQUID with  $A_0/W^2 = 5$ . We stress again that the pattern shown is obtained for large area SQUIDs made of narrow thin-film branches; for reduced areas the interference patterns become more complex, a subject for further study.

Summarising, we have evaluated the field dependence of the maximum supercurrent in narrow edge-type Josephson junctions in thin-film strips; the strip width  $W$  is supposed to be less than both the Pearl length  $\Lambda$  and the thin-film Josephson length  $\ell$  of Eq. (11). Calculations are done in the framework of nonlocal Josephson electrodynamics. We demonstrate that the stray fields cause a pattern  $I_m(H)$  with much reduced distance between zeros,  $\Delta H \sim \phi_0/W^2$ , and with a slowly decreasing maxima in high fields,  $I_m(H) \propto 1/\sqrt{H}$ . The flux dependence of the maximum supercurrent through a SQUID made of narrow thin-film strips with edge-type junctions differs by much from the standard periodicity.

The authors are grateful to J. Mannhart and C. W. Schneider for numerous stimulating discussions. The work of VGK at Ames Laboratory is supported by the Office of Basic Energy Sciences of the U.S. Department of Energy under Contract No. DE-AC02-07CH11358.

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